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ON THE MINIMAL SET OF COMPATIBLES FOR CLOSURE

by

William C. W. Mow

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SUMMARY

Given an incompletely specified sequential switching function in the form of flow table, Paull and Unger had provided a systematic procedure for reducing the function into a set of maximum compatibles. A technique is presented here for selecting a minimal closed set of compatibles without the process of enumeration. The rules have been applied to numerous examples including those presented by Paull and Unger.

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I - INTRODUCTION

For a sequential switching function which is completely specified, the minimization problem has been completely solved by Huffman, Moore and Mealy. The implication here is that an equivalent flow table can always be found with the minimum number of rows. As usual, the rows of a flow table correspond to the internal states and the columns correspond to the inputs. For the class of sequential switching functions which are not completely specified, the flow tables will have blank entries corresponding to the unspecified outputs or the unspecified next state conditions (or both) in the flow table. It was pointed out by Ginsburg that an extension of the classical methods was insufficient to reduce some of the incompletely specified flow tables down to the minimum number of rows possible. If one adheres to the idea that no definite entry is to be made for a blank in the flow table, but to ignore such an output when the machine enters such a total state where the output is unspecified, then the possibility of minimization can be extended. Paull and Unger made use of the above-stated concept in their paper and presented a systematic procedure for finding the set of maximum compatibles. From the maximum compatibles, Paull and Unger resort to the process of enumeration in order to find the minimum closed set of compatibles. The enumeration process entails a trial and error procedure in order to eliminate overlapping states in different maximum compatibles.

In the present paper, a systematic procedure will be given to find the minimum closed set of compatibles without enumeration.

The following section will be a brief review of Paull and Unger's minimization technique.

II Reduction of Incompletely Specified Functions

A brief review of the minimization procedure published by Paull and Unger[1] will be presented in this section.

Given an incompletely specified flow table, the concept of arbitrary assignment for the unspecified output of a total state has allowed additional reductions to be made, as compared with the classical procedures. The following example will illustrate the procedure of reducing an incompletely specified flow table to a set of maximum compatibles.

From the Flow Table IA, an implication table is constructed such as to exhibit all the pairwise compatibles as illustrated in Table IB.

A compatible is defined as a set of states of a flow table that constitutes one member of some closed grouping. The entries in the implication table are made in the following manner.

1) If there is a contradictory pair of outputs for a given row-pair in any input column, then an X is placed in the corresponding cell of the implication table and no further attention need be given to that row-pair. This is true for (13), (35), and (57) in Table IB.

2) If for a row-pair ab the outputs are not contradictory, then enter all row-pairs implied by ab in the ab cell. (Self-implication need not be entered into the table.) Thus in cell (12), (23) is entered; and in cell (15), (26) is entered, etc. In cells (14), (37), (45), (46), and (67), there are no pairs to be entered, so a check mark is inserted.

A second inspection of the implication table is often necessary. If a row-pair ab is an incompatible, then an X is placed in all cells that imply ab, and any other entries in such cells will be ignored. This process is to be repeated until no other incompatible row-pairs are found. A check of the implication Table IB indicates no other modifications to be made; thus it is also the final implication table. All row-pairs with entries which are not X's are compatibles, and those with X's are incompatibles.

	I_1	I_2	I_3
1	2 1	- -	- -
2	3 -	5 -	4 -
3	- 0	1 0	7 -
4	- -	2 -	- -
5	6 -	- 1	4 0
6	4 -	- -	3 -
7	4 -	1 -	- 1

Table IA

2	23					
3	X	47 15				
4	✓	25	12			
5	26	36	X	✓		
6	24	34	37	✓	46 34	
7	24	15 34	✓	12	X	✓
	1	2	3	4	5	6

Table IB

From Table IB the maximum compatibles can be found. A maximum compatible (MC) is defined as a set of rows which form a compatible and which is not included in any larger compatible.

There are two procedures presented by Paull and Unger for finding the MC's, but only one of them will be stated. The procedure is based on the use of incompatible pairs to break down large sets of rows into smaller sets until only MC's remain.

Start with Column 1 of the implication table and work to the right.

1) Write down a set of all states except 1. Then write down a set of states consisting of 1 followed by all states corresponding to non-X rows in Column 1. In our example, we would have at this point (234567) and (124567).

2) Move to the next column containing X's. If ab is an incompatible, then those candidate sets containing both a and b must be split into two sets, one with a omitted and the other with b omitted. The process is repeated through the last column.

In our example the process continues from the candidate sets from Column 1 as follows:

1) (234567) (124567).

2) No change.

3) (24567) (23467) (124567).

4) No change.

5) (2456) (2467) (23467) (12456) (12467).

The maximum compatibles are (23467), (12456) and (12467). The sets (2456) and (2467) are subsets of the others, thus they cannot be MC's.

Subsequent sections are concerned with the method of finding the smallest collection of compatibles without enumeration. By replacing a compatible with a subset of itself, the reduced flow table will, in general, have more unspecified entries. A procedure will be outlined utilizing the set of rules given in Section V to eliminate the overlapping states where possible.

III Maximum Compatible Chart

From a given flow table not all of the derived MC's are needed to insure closure. This section discusses an appropriate selection of the MC's from the MC chart to cover all the states in the flow table. This selection of MC's is not necessarily closed; the discussion of the closure of the chosen sets is postponed until Section V.

In order to select a suitable minimum set to cover all the states from the group of MC's, the maximum compatible chart is constructed in the following manner. In Table II each column carries a number at the top corresponding to one of the states in the flow Table 1A. Each row corresponds to one of the MC's derived in Section II, as identified by the letters A, B and C in the chart. In each row a cross is placed under each internal state contained in the MC represented by that row. Thus in Table II the MC A contains states 2,3,4,6 and 7, so that the crosses in the A row are in the respective columns. The remainder of the chart is completed in a similar manner.

1	2	3	4	5	6	7	
	x	x	x		x	x	A*- (23467)
x	x		x	x	x		B*- (12456)
x	x		x		x	x	C - (12467)

Table II

1	2	3	4	5	6	
x	x	x	x			A*
		x			x	B
				x	x	C
	x			x		D

Table III

An inspection of the MC's indicates that MC's A and B are necessary, since the internal states 3 and 5 are only included in MC's A and B, respectively. This is exhibited very clearly from the MC chart since Columns 3 and 5 have exactly only one cross in them. If a circle is placed around each of the crosses which stand alone in a column, then each row that contains a circle cross will be called an "Essential Maximum Compatible," (EMC). A single asterisk is placed at the end of each row corresponding to an MC which is essential. In this example a check of the EMC's thus indicates a coverage of all the internal states of the flow table. However, if the EMC's do not cover all the internal states, such as in Table III, it is not obvious which of the other combinations of MC's is to be chosen in order to insure closure. This problem is to be discussed in Section V. In any case, one or more choices of the other MC's are needed in order to insure full coverage of internal states; such a choice of MC will be called a "Secondary Maximum Compatible," (SMC).

An interesting configuration of an MC chart arises when a flow table has a set of incompatibles as shown in Table IV A. The pairwise incompatibles are (12), (34) and (56). Thus the MC's can be seen to be (245), (246), (136), (135), (235), (236), (145) and (146). A careful inspection of the MC chart Table IV B shows that there are exactly the same number of crosses in each column, thus making the choice of EMC's and SMC's rather difficult; however, it is quite a relief to find that this type of MC chart only occurs in very special cases of the flow table. The selection of the appropriate compatibles can still be made from an expanded table called the closure table to be defined in Section IV. The MC chart with the above properties will be called a cyclic chart. The necessary and sufficient condition for a chart to be cyclic is that the flow table must have an even number of internal states (m), and the pairwise incompatibles must occur in ascending order - (12), (34) .... (-m).

	1	2	3	4	5
2	X				
3					
4			X		
5					
6					X

Table IV A

	1	2	3	4	5	6	
		X		X		X	(246) - A
		X		X	X		(245) - B
		X	X		X		(235) - C
		X	X			X	(236) - D
	X			X	X		(145) - E
	X			X		X	(146) - F
	X		X			X	(136) - G
	X		X		X		(135) - H

Table IV B

The sufficiency requirement places the following restrictions on the flow table:

1) No other row-pairs may imply an incompatible pair (12), (34) ... (-m). The incompatible pairs may imply any combination of the internal states, since the implied states in this case do not effect the determination of MC's.

2) In any one of the input columns of the flow table there must be only one contradictory pair of output conditions. They must occur in the positions of (12) or

(34) or ... (-m). (Thus all but two of the output entries in any input column must be unspecified.)

The restrictions imply the existence of a lower limit on the number of inputs for a given flow table. If there are more inputs than the lower limit, then all of the output entries in some of the input columns must be unspecified or they must be identical to the output entries in some of the other columns. The examples shown on Tables V A and V B with two and three inputs, respectively, yield the same cyclic maximum compatible chart on Table V C. Note that in Table V B the output entries of input Column I are all unspecified. The output entries in Column I can also coincide with Column I_2 or I_3, and the MC chart would still remain the same.

	I_1	I_2
1	3 1	2 -
2	- 0	1 -
3	1 -	4 0
4	2 -	3 1

Table V A

	I_1	I_2	I_3
1	2 -	- 1	1 -
2	1 -	4 0	2 -
3	- -	1 -	- 0
4	4 -	2 -	3 1

Table V B

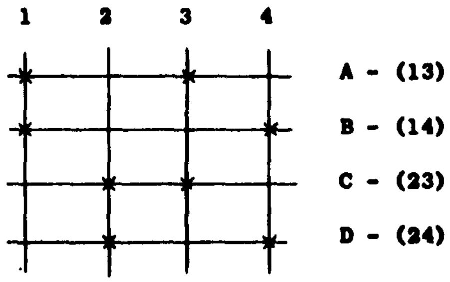

Table V C

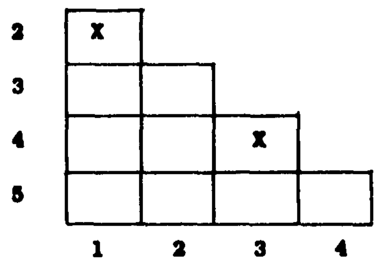

Table VI A

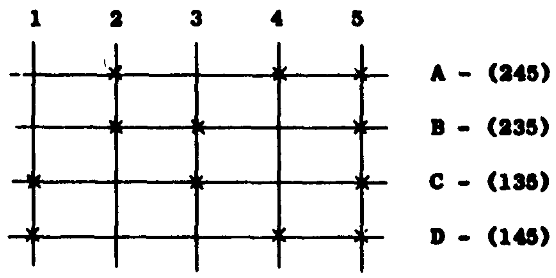

Table VI B

If m is the even number of states and i is the number of inputs, then the lower limit on i is $\frac{m}{2}$ where $m \geq 4$. For $m < 4$, the problem is trivial.

For m odd the above discussion is still valid, except the MC chart will not be cyclic. Depending upon the ascending or decending order of the pair-wise incompatibles, state m or state 1 will be included in all of the maximum compatibles. In Table VI A the implication table of a five-state flow table shows that the pair-wise incompatibles are in the ascending order, (12) and (34), thus the MC chart Table VI B indicates a cross on all of the rows in Column 5. If the pairwise incompatibles are (23) and (45), then all of the MC's will include state 1.

For such cases, except for the end states, the rest of the MC chart is cyclic, thus it will be called a partial cyclic chart.

The lower limit on the number of inputs would be $i = \frac{m-1}{2}$ where $M \geq 5$. For m equal to 1 or 3 no cyclic chart is possible.

The following section investigates the property of closure table from the chosen maximum compatibles.

IV Closure Tables

To facilitate the inspection of closure from the EMC's or the EMC's plus the SMC's, a closure table will be discussed and constructed in this section. The closure table is constructed from all the states of the EMC's or the EMC's plus the SMC's, as the case may be. The selected MC's are called the necessary MC's.

Each of the internal states of a necessary MC is shown explicitly in the form of a row in the closure table, except the outputs have been omitted. The rows corresponding to the internal states of each necessary MC are grouped adjacent to each other.

As an example of how the closure table is constructed, consider Table II. Both Rows A and B have circle crosses, thus MC's A and B are the essential maximum compatibles. In Table VII the associated closure table is shown where the states (23467) and (12456) are shown explicitly and grouped as discussed above. The subsets by implication from the EMC's can be seen directly from the closure table. The implied subsets are (347), (125), and (2346), which are the proper subsets of the essential MC's.

The subsets (347) and (2346) indicate that MC (23467) should not be split up. Therefore, an appropriate closed solution would be (23467) and (125) where the

		I_1	I_2	I_3
A	2	3	5	4
	3	-	1	7
	4	-	2	-
	6	4	-	3
	7	4	1	-
B	1	2	-	-
	2	3	5	4
	4	-	2	-
	5	6	-	4
	6	4	-	3

Table VII

overlapping of states 4 and 6 are eliminated. The solution meets the lower bound of this particular problem. The lower bound of a flow table was defined by Ginsburg[2] to be the number of elements in the largest maximum incompatible.

In the following section a more detailed investigation of the reduction of overlapping states will be given.

V Reduction of Redundant States

The EMC's do not constitute the sufficient condition for closure. For certain flow tables the subsets implied are improper; such nonproper subsets may be avoided by the reduction of overlapping states, or it may be necessary to be introduced for closure. A nonproper subset is an implied subset which is not included in any one of the necessary MC's.

In either case, appropriate rules will be given to suit the requirements of the flow table. The material which follows will be presented with the aid of examples. In Table VIII A the idea of self-linkage will be illustrated. A partial closure table is shown such that the EMC-A implies itself. If there exists a common state between A and some other EMC, k, the common term, cannot possibly be eliminated from the EMC-A.

		I_1	I_2
	2	1 -	3 1
A	3	- 0	5 -
	5	6 -	2 -

Table VIII A

$(35) \longrightarrow (25) \longrightarrow (23)$

Table VIII B

For example, if state 5 is eliminated, (23) implies (35) which indicates 3 and 5 cannot be split. This can be shown nicely as in Table VIII B, where one may start with any pairwise subset of A and it will be closed on itself. The above statement holds true provided that no state in A implies itself in the same input column as the self-implication. The linkage includes all possible combinations of two states out of three.

Before continuing with the discussion, some definitions are required.

Definition 1. - A necessary MC is contained if it is self-implied.

Definition 2. - A necessary MC is truly contained if it is self-implied without any self-implication of states.

Thus Rule I can be stated in the following manner: Rule 1. - If a necessary MC is truly contained, no elimination of common states is possible from that maximum compatible.

Definition 3. - The internal state, i, is primed in an EMC, k, if it is not included in any other necessary MC.

If a necessary MC is contained, the appropriate rule to be applied is Rule III as will be seen.

The self-linkage so far discussed has been through the property of self-implication of the necessary MC in the closure table. An alternate form of self-linkage can take place as illustrated in Table IX A. The partial closure table is constructed such that the common term 3 cannot be eliminated from A without increasing a row member in the final flow table, since EMC (36) implies (34) where state 4 is primed. The process is shown on Table IX B. Thus self-linkage can also be obtained from a coupling of necessary MC's.

		I_1	I_2	I_3
A	3	4	6	3
	4	5	3	-
	5	-	6	-
B	3	4	6	3
	6	-	-	4

Table IX A

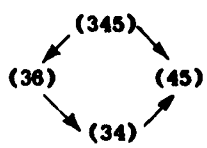

Table IX B

In general, the self-linkage of a necessary MC produced by a coupling with another MC can be divided into two classes.

1) The common term or terms in a MC, k, cannot be eliminated if every non-primed state of k forms a compatible with a primed state of k in the linkage without the aid of self-implication of the non-primed state, as shown in Table IX B.

2) Provided no other necessary MC implies a compatible formed by the non-primed state or states with a primed state of k, the non-primed states may be eliminated if the linkage is formed by:

a) The subsets of MC, k, not involving the non-primed state or states.

b) The compatibles formed with the aid of self-implication of the non-primed state or states of MC, k.

c) Any other necessary MC's or subsets which may or may not include the non-primed state or states of MC, k.

d) Any combination of a, b and c.

Definition 4. - For a non-contained MC, k, if a self-linkage is formed through the coupling of other necessary MC's such that every non-primed state of k forms a compatible with a primed state of k in the linkage (without self-implication of the non-primed state), k is known as a fundamental MC.

Rule II - The non-primed state or states cannot be eliminated from a fundamental MC.

If a necessary maximum compatible is neither fundamental nor truly contained, then Rule III can be applied appropriately.

Rule III - If an MC, k, is neither truly contained nor fundamental, then the non-primed states, l_{i1}, l_{i2} ... l_{im}, of k may be eliminated if and only if the following are valid:

1) The implied compatible (l_{i1} ... l_{im}, l_{j1} ... l_{jn}) is derived with the aid of self-implication of l_{i1} ... l_{im} where l_j's are the prime states of k.

2) The compatible ($l_{i1} \ldots l_{im}$, $l_{j1} \ldots l_{jn}$) is not implied by any EMC or by subsets of any EMC, including k, whose constituents are all primed.

3) The compatible (l_i's, l_j's) is not implied by any other compatible ($\bar{l}_i$'s, $\bar{l}_j$'s) which is in turn implied by the subsets whose constituents are all primed.

4) The compatible (l_i, l_j) is not implied by any essential pairwise compatible. (Pairwise compatibles are to be discussed in Rule IV.)

The partial closure table shown on Table X A indicates that EMC-A is contained. The internal state 3 in MC, A, is self-implied, but 3 cannot be eliminated from A since (34) implies (36) where compatible (34) is implied by the primed states of A.

		I_1	I_2
A	2	6	3
	3	3	-
	6	2	4
B	3	3	-
	4	6	5

Table X A

		I_1	I_2
A	2	6	3
	3	3	-
	6	2	4
B	3	1	-
	4	5	6

Table X B

Thus (36) must be a proper subset of (236).

In Table X B the partial closure table shows that state 3 may be eliminated from EMC-A provided (15) does not imply any proper subset of (236) which includes the non-primed state 3. For an illustration of a non-fundamental self-linked EMC, see Table XI A. The non-primed state 3 can be eliminated from EMC-(345); however, MC(36) must remain unchanged since (36) is implied by the primed states of A. The EMC'- (345) is non-fundamental as can be seen from Table XI B, where 3 does not form a compatible with either 4 or 5 of MC, A.

		I_1	I_2	I_3
A	3	4	6	5
	4	5	3	-
	5	-	6	-
B	3	4	6	5
	6	-	-	4

Table XI A

$$
\begin{tikzcd}
 & (345) \arrow[dl] \arrow[dr] & \\
(36) & & (45) \arrow[ll, bend left]
\end{tikzcd}
$$

Table XI B

Rule IV - The elimination of a common state shared by pairwise compatibles is possible if the following is true:

1) The pairwise compatible is not implied by:

a) Any truly contained EMC,

b) Any subset of a contained EMC which is truly contained,

c) Any fundamental MC,

d) Any EMC or subsets of EMC whose constituents are all primed.

2) The pairwise compatible is not implied by any other pairwise compatible which is in turn implied by the above-mentioned sets.

3) The only implication of the pairwise compatible is by itself.

It had been stated in the beginning of this section that the EMC's do not always form a closed solution. The subsets implied by the EMC's may not be included in any of the EMC's themselves. In order to determine whether the extraneous subsets are needed for closure, one of such rules to be applied can be stated in the following manner.

Rule V - If a non-proper subset has been implied by the EMC, k, then k should be checked for self-linkage. If k is self-linked with the aid of the improper subset, then the non-proper subset must be introduced for closure. Otherwise, the non-proper subset may be avoided by the elimination of overlapping states. Rule V is a sufficient condition for the inclusion of the non-proper subset.

In the following example, Rule V will be illustrated. From the flow table as shown in Table XII A, the MC's are found to be (1346), (1234) and (3456). In Table XII B an MC chart has been constructed which indicates the MC's (1234) and (3456) are the essentials. From the closure table shown on Table XII C the subsets implied are (16) and (45). Since (16) is non-proper and implied by EMC-C, the EMC-C must be checked for self-linkage by Rule V.

$$\begin{tikzcd} & (3456) \arrow[dl] \arrow[dr] & \\ (16) & & (45) \arrow[ll, bend left] \end{tikzcd}$$

	I_1	I_2
1	- 0	- -
2	- 0	1 -
3	- -	- 0
4	1 -	-
5	6 1	4 -
6	-	5 -

Table XII A

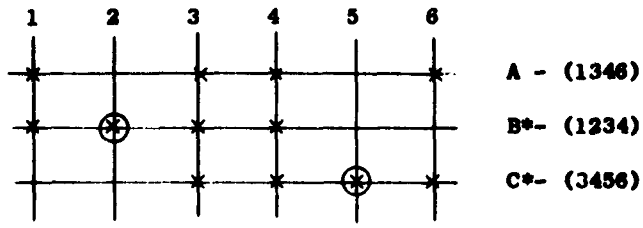

Table XII B

		I_1	I_2
B	1	-	-
	2	-	1
	3	-	-
	4	1	-
C	3	-	-
	4	1	-
	5	6	4
	6	-	5

Table XII C

	I_1	I_2
B	- 0	D 0
C	D 1	C -
D	- 0	C -

Table XII D

	I_1	I_2
B'	B' 0	B' 0
C'	D' 1	B' -
D'	-	C' -

Table XII E

The EMC-C is self-linked with the aid of the non-proper subset (16), thus (16) must be included in the solution for closure. By applying Rule III to the necessary MC's (16), B, and C, states 3 and 6 may be eliminated from MC, C; and states 1 and 4 may be eliminated from MC, B. Thus the final solution for closure is (16), (23) and (45). If the compatible (16) is labeled D and (23) and (45) are respectively B and C, then the final flow table is shown in Table XII D.

An alternate procedure for determining whether a non-proper subset implied by the EMC's is needed is to investigate the subsets implied by the primed states of those EMC's.

Definition 4 - For a compatible which is neither truly contained nor fundamental such that none of the states can be split without breaching an implication, then that compatible is bounded.

Rule VI - The non-proper subsets are to be introduced if it is implied by a bounded compatible.

Rule VI suggests a procedure to be followed in certain cases where Rule V is not so obvious, such as a flow table with many input columns. If a non-proper subset is implied by a necessary MC, K, which is neither truly contained nor fundamental, then one may begin the investigation by observing the subsets implied by the primed states of K. If the compatible, k, formed by the primed states of K is bounded (which could be true if it is implied either by a truly contained or a fundamental MC), then the improper subset implied directly by k or by subsets implied by k which in turn implies the improper subset must be included for closure. However, if the compatible, k, is not bounded, then the improper subset implied by k directly or indirectly as mentioned above may be avoided by splitting the compatible, k. The exact procedure is to be outlined in the next section; however, a brief example will be done here to illustrate the above-mentioned idea.

Again consider Table XII; the improper subset is implied by MC, C. The compatible formed by the primed states of MC, C, is not bounded. Therefore, Rule VI indicates that the improper subset implied in the chain (56) → (45) → (16) may be broken up if 5 and 6 are split.

Therefore, the final solution in this case is (1234), (5) and (6), which is respectively assigned the letters B', C' and D' as shown on Table XII E.

Rule VII - Non-proper subsets must be introduced for closure if it is implied by:

1) Any truly contained EMC,

2) Any subsets of contained EMC's which are truly contained,

3) Any fundamental MC,

4) Any subsets which are in turn implied by any of the above sets.

Thus it should be obvious that non-proper subsets implied by pairwise compatibles may be avoided if the pairwise compatibles are not implied by any of the conditions of Rules VI and VII.

If from the MC chart the EMC's do not cover all the states of the flow table, then one can do either of the following two things:

1) Construct a closure table for all of the MC's from which a secondary MC may be chosen to cover all the states, then apply the given rules,

2) Select a secondary MC which with the EMC has a minimum of overlapping, and construct the accompanying closure table, then apply the rules.

In Table XIII A a flow table is constructed such that the essential maximum compatibles do not cover all the internal states. The MC chart, Table XIII B, lists the MC's and shows that MC, A, is essential. For a minimum of overlapping states, the SMC will be chosen to be MC, C. By this choice only the internal state 2 is present in both the EMC and SMC. In Table XIII C the closure table is constructed for the chosen EMC and SMC.

The subsets by implication are (35), (24), (34) and (123). But (35) is a non-proper subset implied by both of the necessary MC's. Thus the EMC and the SMC must both be checked for self-linkage since both of the necessary MC's implied the non-proper subset (35). If either one of the necessary MC's is self-linked with the aid of the non-proper subset, the non-proper subset must be introduced for closure.

$$\begin{tikzcd} & \begin{matrix}\text{A}\\(1234)\end{matrix} \arrow[dl] \arrow[dr] & \\ (35) & & (24) \arrow[ll, bend left] \end{tikzcd}$$

and

$$\begin{tikzcd} & \begin{matrix}\text{C}\\(256)\end{matrix} \arrow[dl] \arrow[dr] & \\ (123) \arrow[dr] & & (35) \\ & (24) \arrow[ur] & \end{tikzcd}$$

	I_1	I_2
1	- 0	2 -
2	3 -	- -
3	- -	4 0
4	5 0	- 0
5	1 1	3 -
6	2 -	5 1

Table XIII A

	1	2	3	4	5	6
A*- (1234)	*	*	*	*		
B - (235)		*	*		*	
C - (256)		*			*	*
D - (126)	*	*				*

Table XIII B

		I_1	I_2
	1	-	2
	2	3	-
A* -	3	-	4
	4	5	-
	2	3	-
C -	5	1	3
	6	2	5

Table XIII C

	I_1	I_2
A	B' 0	A 0
B'	A 1	A 0
C	A -	B' 1

Table XIII D

It can be seen that EMC-A is fundamental and self-linked with the aid of the non-proper subset (35). Thus the common term (2) cannot be eliminated from EMC-A, and (35) must be introduced for closure by Rules II and V, respectively. The problem now is to reduce the overlapping states from the compatibles (1234), (35) and (256). The SMC, C, is shown to be non-fundamental, thus the overlapping states 2 and 5 may be both eliminated by Rules II and III. The final solution is $\overset{A}{(1234)}$, $\overset{C}{(6)}$ and $\overset{B'}{(35)}$ as shown on Table XIII D.

For partial cyclic and cyclic charts the closure table should consist of all the MC's. The given rules are applicable to any combination of MC's which covers all the states of the flow table. Following is an example of a cyclic table. The flow table is shown on Table XIV A. From the location of the incompatible in the implication table, one automatically draws the conclusion of a cyclic maximum compatible chart. The MC's are (13), (14), (23) and (24) which are shown in Table XIV C. Two obvious candidate sets which will cover all the states are MC's B and C and MC's A and D.

Let's consider the candidate set with MC's B and C. The subset implied is (13), but it is non-proper. Therefore, MC B which implies (13) must be checked for self-linkage. It is obvious that the compatible (14) is neither bounded nor self-linked, thus with the application of either Rule VI or Rule VII, respectively, the final solution for closure is (1), (4) and (23). Similarly, the set consisting of MC's A and D may be synthesized to be (2), (4) and (13).

	I_1	I_2
1	- 1	1 -
2	1 0	2 -
3	- -	- 0
4	4 -	3 1

Table XIV A

2	X		
3	✓	✓	
4	13	14 23	X
	1	2	3

Table XIV B

		I_1	I_2
A	1	-	1
	3	-	-
B	1	-	1
	4	4	3
C	2	1	2
	3	-	-
D	2	1	2
	4	4	3

Table XIV C

The cyclic MC chart derived from a flow table of many internal states may have many candidate sets for closure. The proper sets to be chosen can usually be seen from the closure table, since for the cyclic MC chart the minimum number of maximum compatibles needed to cover all of the states is two. However, the set of MC's chosen to cover all the states should not imply any non-proper subsets. (Otherwise, it would mean an extra row member in the final flow table.) Once the candidate sets are chosen, the procedure is no different from any other non-cyclic cases.

It has been seen in this section that the procedure for finding the minimal closed set of solutions does not necessarily make use of all the rules. However, it should be mentioned that the rules given can be applied to all classes of MC charts. Indeed for the cyclic MC chart the procedure may be quite long; however, one does not often encounter such a situation. In the following section a systematic procedure will be hopefully outlined and examples will be shown.

VI Procedure

From the previous section it can be seen that the rules given are sufficient to eliminate the redundant states where possible. However, a more systematic procedure is needed for the application of the rules. In general, the procedure to be discussed can be divided into two parts.

A) The necessary maximum compatibles imply only proper subsets.

B) The necessary maximum compatibles imply non-proper subsets in addition to the proper subsets.

In both cases one should begin the problem from the maximum compatibles found from the implication table (as defined by Paull and Unger[1]).

Step 1. Construct the maximum compatible chart.

Step 2. Choose the appropriate set of EMC's or the EMC's plus the SMC's so that it covers all the states of the flow table, and construct the corresponding closure table. For a cyclic maximum compatible chart it is preferable to list all the maximum compatibles in the closure table from which the candidate sets may be exhibited.

Step 3. Determine whether the subset implied by any of the necessary maximum compatibles is proper or non-proper.

Step 4. Determine from each necessary maximum compatible with 3 or more states whether it is truly contained or fundamental. (Rules I and II.) (The fundamental property is sufficient, but as mentioned previously, it is not necessary.)

The procedure which follows will apply subscripts A and B to the class of subsets which are all proper and to the class of subsets which includes non-proper subsets, respectively.

Step 5A.

a) Apply Rules III and IV to the necessary MC's which are neither truly contained nor fundamental to eliminate the overlapping states. Repeat the above rules if necessary.

b) If none of the necessary MC's are truly contained or fundamental, then apply Rule III to the sets implied by the primed states of the necessary MC's. Associate the sets implied with the primed states wherever possible. Continue the above procedure until all subsets implied are proper. (See Example 2 in Section VII.)

Step 5B.

a) For each necessary MC which implies non-proper subsets, if either of the cases of Step 4 holds, then apply Rule VII.

b) If the truly contained or fundamental MC's do not imply the non-proper subset, apply Rule III or IV, and Rule V or VI to the other necessary MC's. Repeat if necessary.

c) If none of the EMC's are truly contained or fundamental, apply Rules III and VI to the sets implied by the primed states of the EMC's. Continue the process until all sets implied are proper and non proper sets eliminated if possible. Otherwise the non-proper subset must be included. (See Example 3.)

For the sake of clarity Step 6 will be postponed until after the following example. From Table XV A the MC's are (245), (367), (127) and (46), respectively, given the letters A, B, C, and D. The EMC's are shown in Table XV B to be A, B, and C. Correspondingly, the closure table is shown on Table XV C, which brings the procedure up to Step 3.

Step 3.

Subsets implied: (37), (46), (245), (127) and (24).

(46) is non-proper.

	I_1	I_2
1	2 1	- 0
2	7 1	4 -
3	- 0	4 0
4	- -	- 1
5	3 1	6 -
6	- 0	5 -
7	1 -	2 0

Table XV A

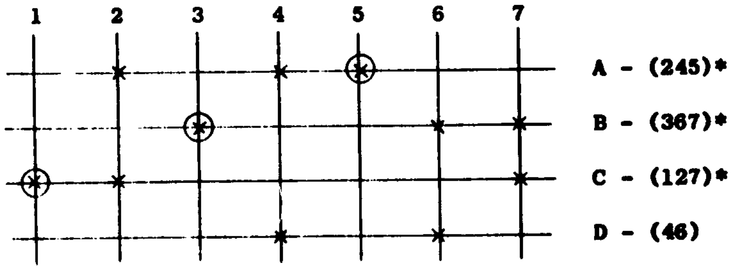

Table XV B

		I_1	I_2
A	2	7	4
	4	-	-
	5	3	6
B	3	-	4
	6	-	5
	7	1	2
C	1	2	-
	2	7	4
	7	1	2

Table XV C

	I_1	I_2
A'	B' 1	D' 1
B'	C' 0	A' 0
C'	A' 1	- 0
D'	- 0	A' 1

Table XV D

Step 4.

From the closure table it is obvious that the EMC-C is truly contained and EMC's A and B are neither truly contained nor fundamental. Therefore, the overlapping states 2 and 7 cannot be eliminated from the maximum compatible C.

Step 5B-b

EMC-C implies (24); therefore A-(245) should remain intact. Since A is bounded, (46) must be introduced as a non-proper subset, and (367) should not be split since (37) is implied by A also.

It seems that we need all of the MC's to insure closure. However, if we apply Rule III to the truly contained MC at this stage, we find that both of the overlapping states 2 and 7 may be eliminated since no other EMC implies either 2 or 7 with 1. Therefore, by eliminating 2 and 7 from MC-C and 6 from MC-B, the final solution becomes (1), (46), (37) and (245). This solution has only one overlapping state. The final flow table is shown in Table XV D for the compatibles labeled C', D', B' and A', respectively. It should be clear that Step 6, to be given subsequently, will only apply if at the end of Step 5 two or more overlapping states exist between the truly contained MC and the other necessary compatibles. If there is only one such overlapping state, then it cannot be eliminated from the truly contained MC, and the process ends before it reaches Step 6.

Step 6

Apply Rule III to the truly contained maximum compatible if there exists two or more common states between the truly contained MC and the other necessary MC's at the end of Step 5. Then repeat Step 5 if necessary.

For an illustration of a flow table without the necessity of Step 6, the flow table shown in Table XVI A yields the same set of MC's as Table XV B. The maximum compatible chart will therefore be omitted in this case. Again the essential MC's are (245), (367) and (127), labeled A, B, and C, respectively.

	I_1	I_2
1	2 1	- 0
2	7 1	4 -
3	- 0	4 0
4	7 -	- 1
5	3 1	6 -
6	- 0	5 -
7	1 -	5 0

Table XVI A

		I_1	I_2
A	2	7	4
	4	7	-
	5	3	6
B	3	-	4
	6	-	5
	7	1	5
C	1	2	-
	2	7	4
	7	1	5

Table XVI B

	I_1	I_2
A'	B' 1	B' 1
B'	C' 0	A' 0
C'	C' 1	A' 0

Table XVI C

Step 3.

Subsets implied: (37), (46), (45) and (127). (46) is improper.

Step 4

(127) is truly contained.

Step 5B-b

MC (127) implies (45); therefore (45) is a bounded compatible. However, A(245) is not bounded since 2 is not implied by any other maximum compatible with either 4 or 5. Therefore, by Rule VI the non-proper subset (46) may be avoided by omitting 2 from (245). By Rule III the overlapping state 7 cannot be eliminated from MC (367) since (45) implies (37). Therefore, the final solution is (45) (367) and (127) as shown on Table XVI C.

VII Examples

The examples provided in this section are mainly designed to illustrate the Step 5A-b and Step 5B-c, since in Section VI illustrations have been provided for the other steps. However, Example I is included to demonstrate again the ideas of Rules I and II.

Example I

From Table XVII A the maximum compatibles are found to be (1267), (345) and (25) which are designated as A, B, and C, respectively, in the MC chart of Table XVII B. It can be seen that the EMC's are A and B, which also cover all the states of the flow table. The closure table with EMC's A and B is constructed as shown on Table XVII C. This brings us to Step 3.

Step 3.

Subsets implied: (267), (126) and (345). All subsets are proper.

Step 4.

a) (345) implies (534), respectively

b)

$$\begin{tikzcd} & (1267) \arrow[dl] \arrow[dr] & \\ (267) & & (126) \arrow[ll, bend left] \end{tikzcd}$$

EMC - (1267) Fundamental.

EMC - (345) Truly contained.

Thus the process terminates since no states can be eliminated from the MC's by Rules I and II. This is obviously true since all the states in both of the MC's are all primed. This is indeed a trivial example. The final table is shown on Table XVII D.

	I_1	I_2	I_3
1	7 1	- -	6 0
2	2 -	- -	2 0
3	3 0	5 1	6 1
4	4 0	3 1	7 1
5	- 0	4 1	2 -
6	6 1	- -	2 0
7	6 1	7 0	1 -

Table XVII A

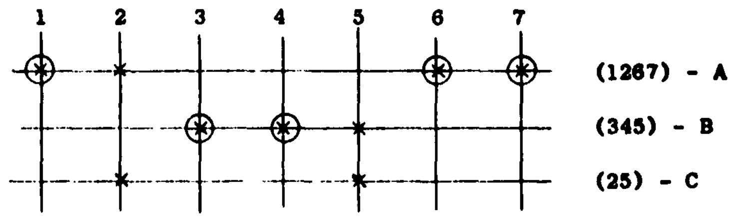

Table XVII B

		I_1	I_2	I_3
A	1	7	-	6
	2	2	-	2
	6	6	-	2
	7	6	7	1
B	3	3	5	6
	4	4	3	7
	5	-	4	2

Table XVII C

	I_1	I_2	I_3
A	A 1	A 0	A 0
B	B 0	B 1	A 1

Table XVII D

Example 2

The closure table shown on Table VII, Section IV, will be continued here to find the minimal closed set.

Step 3.

Subsets implied: (125), (347) and (2346). All subsets are proper.

Step 4.

Neither A nor B is truly contained.

Step 5A-b.

Subsets formed by the primed states of A and B are:

a) (37) $\quad \begin{tikzcd}[row sep=small] (15) \arrow[d] \\ (26) \end{tikzcd}$ ← Implication

b) Application of Rule III

$$\begin{tikzcd}[column sep=small] & (2367) \arrow[dl] \arrow[d] \arrow[dr] & & (15) \arrow[d] \\ (34) & \cancel{(15)} & (347) & \cancel{(26)} \end{tikzcd}$$ ← Implication

c)
$$\begin{tikzcd}[column sep=small] & (23467) \arrow[dl] \arrow[d] \arrow[dr] & & (15) \arrow[d] \\ \cancel{(34)} & (125) & \cancel{(341)} & (26) \end{tikzcd}$$ ← Implication

d) (23467) (125)

Thus in Step 5A-b a repeated application of Rule III is sufficient to find the minimal closed solution. The compatibles shown crossed out are already proper subsets of that step.

Example 3

From the flow table of Table XVIII A the maximum compatibles are found as shown on Table XVIII B. By inspection MC's A and C are essential and cover all the states of the flow table.

Step 3.

Subsets implied: (1279), (5679), (358), (1379) and (457). Subsets (1279) and (358) are non-proper.

Step 4.

EMC's A and C are noncontained.

Step 5B-c

Subsets formed by the primed states of EMC:

a)
$$\begin{tikzcd}[column sep=small] & (248) \arrow[dl] \arrow[dr] & & & (369) \arrow[dl] \arrow[d] \arrow[dr] & \\ (\underline{279}) & & (569) & (45) & (137) & (\underline{38}) \end{tikzcd}$$ ←Implications

Where underlined compatibles are non-proper, apply Rule VI. Since (248) and (369) are not bounded, one may try to eliminate the non-proper subsets by omitting one of the states shown above and beginning the process again.

b) (24) → (79), (56)   (69) → (13)   ← Implications

By Rule III, (c) is as follows:

c) (24) → ~~(79)~~, ~~(56)~~   (1356 7 9) → ~~(1379)~~, (457), <u>(38)</u>   ← Implications

By Rule III, (d) is as follows:

d) (2457) → ~~(179)~~, ~~(567)~~, ~~(36)~~   (135679) → (Completed)   <u>(38)</u> → ~~(27)~~, ~~(59)~~   ← Implication

e) A' (2457)   B' (135679)   C' <u>(38)</u>   Final Solution.

The compatibles shown crossed out are already proper subsets up to that step. Solution is shown on Table XVIII D.

	I_1	I_2	I_3	I_4
1	9 0	5 0	8 0	-
2	9 0	5 -	5 -	2,1
3	7 0	5 -	8 -	3,0
4	7 -	6 -	- 0	2 1
5	7 0	- 0	-	-
6	3 -	- 0	3 -	3,0
7	1 -	7 0	3 0	-
8	2 0	9 0	- 0	-
9	1 0	4 0	3 0	-

Table XVIII A

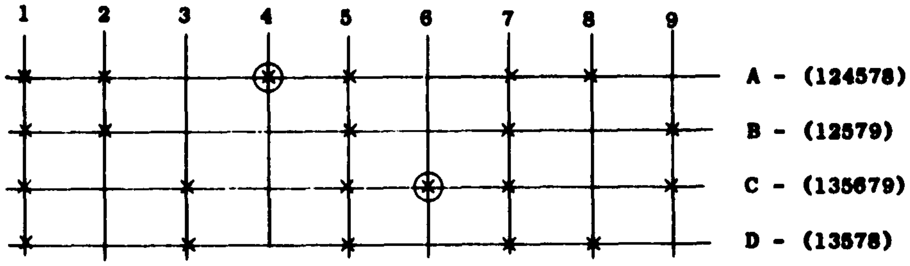

Table XVIII B

	I_1	I_2	I_3	I_4
1	9	5	8	-
2	9	5	5	2
4	7	6	-	2
5	7	-	-	-
7	1	7	3	-
8	2	9	-	-
1	9	5	8	-
3	7	5	8	3
5	7	-	-	-
6	3	-	3	3
7	1	7	3	-
9	1	4	3	-

Table XVIII C

	I_1	I_2	I_3	I_4
A'	B' 0	B' 0	B' 0	A' 1
B'	B' 0	A' 0	C' 0	B' C' 0
C'	A' 0	B' 0	C' 0	B' C' 0

Table XVIII D

VIII Conclusion

From the maximum compatibles derived from any incompletely specified flow table, the problem of choosing a minimal set of compatibles has been investigated. A set of rules and definitions has been given and defined such that hopefully it might cover most of the cases which are likely to arise. The procedure does not yield all of the possible solutions, and it does not always meet the lower bound defined by Ginsberg[2]. However, for certain flow tables the lower bound is not necessarily obtainable. For flow tables which do not have any EMC's, a systematic approach remains to be found to start the application of the rules.

In conclusion, the procedure does indicate an approach to the problem for those flow tables which have EMC's and SMC's.

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